

CHSH violation and entropy – concurrence plane

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We characterize violation of CHSH inequalities for mixed two-qubit states by their mixedness and entanglement. The class of states that have maximum degree of CHSH violation for a given linear entropy is also constructed.

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I. INTRODUCTION

Physically allowed degree of entanglement and mixture for two - qubit mixed states were investigated by Munro *et al.* in terms of concurrence $C(\rho)$ and normalized linear entropy $S_L(\rho)$ [1]. These authors characterize the subset Λ on (C, S_L) plane corresponding to possible states of the system and in particular identify maximally entangled mixed states ρ_{MEMS} laying on the boundary of that subset. The states ρ_{MEMS} have maximal allowed entanglement for a given degree of mixedness. They obtained this result analytically for some class of states. Numerical results suggests correctness of the picture for general two-qubit states.

In this note, we consider the problem of violation of Bell – CHSH inequalities [2, 3] for mixed states. It is well known that quantum states violating these inequalities have to be entangled [4, 5], but on the other hand, CHSH violation is not necessary for mixed state entanglement [6]. In the context of the results of Ref.[1], we address the following question: what are the subsets of Λ which correspond to states violating Bell – CHSH inequalities? In our previous publication [7], we have studied the structure of such subsets in the case of specific class of quantum states. The results show that Λ is a sum of disjoint subsets Λ_V , Λ_{NV} and Λ_0 with the following properties: states belonging to Λ_V violate CHSH inequalities, whereas states from Λ_{NV} fulfil all CHSH inequalities. The subset Λ_0 has somehow unexpected property: for any pair $(S_L, C) \in \Lambda_0$ there are two families of states with the same entropy and concurrence such that all states from one family violate CHSH and at the same time, all states from the other family fulfil all CHSH inequalities. In the present paper, we continue these investigations for general class of two-qubit states. First we consider larger class of states still admitting explicit formulas for linear entropy, concurrence and degree of CHSH violation. Unfortunately, analytical analysis of the relation between these functions is not possible. Numerical investigations lead to some modifications of the picture from Ref. [7], but the general structure is not changed. Finally, this problem is studied using numerically generated arbitrary density matrices. The results indicate that the structure of Λ for

general two-qubit states seems to be the same.

We consider also the problem of maximal violation of CHSH inequalities. For a class of mixed states we obtain counterpart of the result of Ref.[1], namely we find the form of mixed states with maximal degree of CHSH violation for given linear entropy. All that states lie on specific curve on entropy – concurrence plane. Numerical results suggest also that general two – qubits states with this property satisfy the same relation between entropy and concurrence. As we show, maximal violation of Bell inequalities with fixed linear entropy is not equivalent to the maximal entanglement under the same conditions.

II. CHSH INEQUALITIES

Let $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$ be the unit vectors in \mathbb{R}^3 and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. Consider the family of operators on two - qubits Hilbert space $\mathcal{H}_{AB} = \mathbb{C}^4$

$$B_{CHSH} = \mathbf{a} \cdot \boldsymbol{\sigma} \otimes (\mathbf{b} + \mathbf{b}') \cdot \boldsymbol{\sigma} + \mathbf{a}' \cdot \boldsymbol{\sigma} \otimes (\mathbf{b} - \mathbf{b}') \cdot \boldsymbol{\sigma} \quad (\text{II.1})$$

Then Bell - CHSH [3] inequalities are

$$|\text{tr}(\rho B_{CHSH})| \leq 2 \quad (\text{II.2})$$

If the above inequality is not satisfied by the state ρ for some choice of $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$, we say that ρ violates Bell-CHSH inequalities. In the case of two-qubits, the violation of Bell - CHSH inequalities by mixed states can be studied using simple necessary and sufficient condition [8, 9]. Consider real matrix

$$T_\rho = (t_{nm}), \quad t_{nm} = \text{tr}(\rho \sigma_n \otimes \sigma_m) \quad (\text{II.3})$$

and real symmetric matrix

$$U_\rho = T_\rho^T T_\rho \quad (\text{II.4})$$

where T_ρ^T is the transposition of T_ρ . Let

$$m(\rho) = \max_{j < k} (u_j + u_k) \quad (\text{II.5})$$

and $u_j, j = 1, 2, 3$ are the eigenvalues of U_ρ . As was shown in [8, 9]

$$\max_{B_{CHSH}} \text{tr}(\rho B_{CHSH}) = 2 \sqrt{m(\rho)} \quad (\text{II.6})$$

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Thus (II.2) is violated by some choice of a, a', b, b' if and only if $m(\rho) > 1$.

We need also the measures of degree of entanglement and mixture for given state. In the case of two qubits, the useful measure of degree of entanglement is concurrence $C(\rho)$

$$C(\rho) = \max(0, 2\lambda_{\max}(\hat{\rho}) - \text{tr}\hat{\rho}) \quad (\text{II.7})$$

where $\lambda_{\max}(\hat{\rho})$ is the maximal eigenvalue of $\hat{\rho}$ and

$$\hat{\rho} = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}, \quad \tilde{\rho} = (\sigma_2 \otimes \sigma_2)\bar{\rho}(\sigma_2 \otimes \sigma_2)$$

with $\bar{\rho}$ denoting complex conjugation of the matrix ρ . It is known that $C(\rho)$ can be used to obtain entanglement of formation, which is natural measure of entanglement for mixed states [10, 11]. To measure degree of mixture, or deviation from pure state, we use linear entropy

$$S_L(\rho) = \frac{4}{3}(1 - \text{tr}\rho^2) \quad (\text{II.8})$$

which is normalized such that its maximal value equals 1.

III. CHSH INEQUALITIES AND ENTROPY – CONCURRENCE PLANE

To study the subset of entropy – concurrence plane corresponding to violation of CHSH inequalities, consider first the following class \mathcal{E}_0 of states

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & \frac{1}{2}ce^{i\theta} & 0 \\ 0 & \frac{1}{2}ce^{-i\theta} & b & 0 \\ 0 & 0 & 0 & 1-a-b \end{pmatrix} \quad (\text{III.1})$$

where

$$c \in [0, 1], a, b \geq 0, \theta \in [0, 2\pi]$$

and

$$ab \geq \frac{c^2}{4}, \quad a+b \leq 1$$

Notice that for $\rho \in \mathcal{E}_0$

$$C(\rho) = c$$

Define the subset $\Lambda \subset \mathbb{R}^2$

$$\Lambda = \{(S_L(\rho), C(\rho)) : C(\rho) > 0 \text{ and } \rho \in \mathcal{E}_0\} \quad (\text{III.2})$$

In the paper [7], we analysed the set (III.2) and we have shown that it is a sum of disjoint subsets Λ_V , Λ_0 and Λ_{NV} with properties:

1. If $(s, c) \in \Lambda_V$, then every state $\rho \in \mathcal{E}_0$ such that $S_L(\rho) = s$ and $C(\rho) = c$ satisfies $m(\rho) > 1$.

2. If $(s, c) \in \Lambda_0$, then there exist states $\rho_1, \rho_2 \in \mathcal{E}_0$ such that

$$S_L(\rho_1) = S_L(\rho_2) = s, \quad C(\rho_1) = C(\rho_2) = c$$

and $m(\rho_1) > 1$, but $m(\rho_2) < 1$.

3. If $(s, c) \in \Lambda_{NV}$, then every state $\rho \in \mathcal{E}_0$ such that $S_L(\rho) = s$ and $C(\rho) = c$ satisfies $m(\rho) < 1$.

Detailed analytic description of regions Λ_V , Λ_0 and Λ_{NV} as well as the proof of the above properties, can be found in Ref. [7].

In the present paper, we try to extend this analysis to the larger class of two-qubit states. For general mixed two-qubit states there is a bound on concurrence that guarantees violation of CHSH inequalities irrespective of linear entropy. It follows from the result of Verstraete and Wolf [12] that minimal violation of CHSH inequality for given concurrence C is equal to

$$m_{\min} = \max(1, 2C^2) \quad (\text{III.3})$$

So if

$$C(\rho) > \frac{1}{\sqrt{2}} \quad (\text{III.4})$$

then minimal value of m is greater than 1, and every state satisfying (III.4) violates CHSH inequality. On the other hand, there is a bound on linear entropy that guarantees fulfilling CHSH inequalities irrespective of concurrence. It is given by the result of Santos [13] that all states with (normalized) linear entropy

$$S_L(\rho) > \frac{2}{3} \quad (\text{III.5})$$

satisfy all CHSH inequalities. We see that these general bounds are compatible with our previous analysis (FIG. 1) and possible modifications can occur in region Λ_V below the line $C = \frac{1}{\sqrt{2}}$ and in region Λ_{NV} on the left hand side of the line $s = \frac{2}{3}$. Consider now the larger class \mathcal{E}_1 of states of the form

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix} \quad (\text{III.6})$$

One can check that for that class

$$C(\rho) = \max(0, C_1, C_2) \quad (\text{III.7})$$

where

$$\begin{aligned} C_1 &= 2(|\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}) \\ C_2 &= 2(|\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}) \end{aligned} \quad (\text{III.8})$$

and

$$S_L(\rho) = 1 - \rho_{11}^2 - \rho_{22}^2 - \rho_{33}^2 - \rho_{44}^2 - 2|\rho_{14}|^2 - 2|\rho_{23}|^2 \quad (\text{III.9})$$

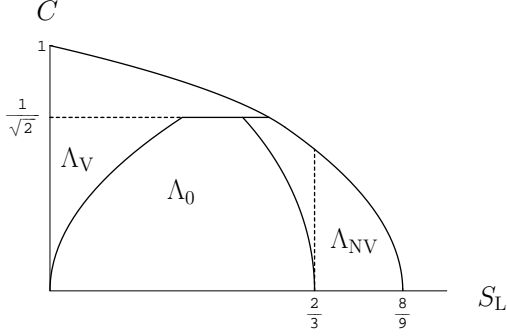


FIG. 1: The structure of the set Λ and Verstraete – Wolf and Santos bounds (dotted lines)

Moreover,

$$m(\rho) = \max [4(|\rho_{41}| + |\rho_{23}|)^2, 4(|\rho_{41}| - |\rho_{23}|)^2, (\rho_{11} - \rho_{22} - \rho_{33} + \rho_{44})^2] \quad (\text{III.10})$$

For the class (III.6), analytic description of regions of Λ where $m(\rho) > 1$ or $m(\rho) < 1$ is not possible, but it can be done numerically. Results of numerical analysis of the class \mathcal{E}_1 are presented on FIG. 2. We see that the region Λ_V where all states violate CHSH inequalities is not changed, but there are states with $m(\rho) > 1$ for some points in Λ_{NV} , so the region Λ_0 is slightly enlarged. To study this problem for general

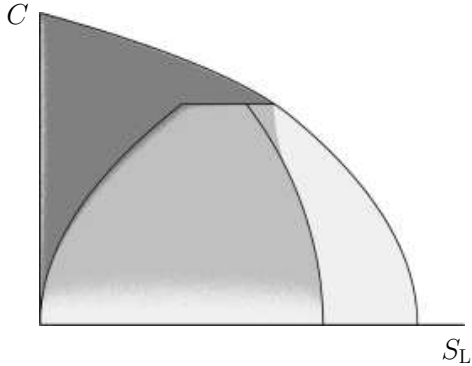


FIG. 2: Numerical analysis of the set Λ for the class \mathcal{E}_1 : $m(\rho) > 1$ (dark grey), $m(\rho) > 1$ and $m(\rho) < 1$ (grey), $m(\rho) < 1$ (light grey)

two-qubit density matrices, we numerically generate $3 \cdot 10^6$ randomly chosen density matrices. For such two-qubit states the structure of the set of pairs (S_L, C) is very simple. There are only points corresponding to $m(\rho) > 1$ or $m(\rho) < 1$ (FIG. 3). Notice that by the method of random choice of states, not all points of Λ are achieved (the boundary of generated set correspond exactly to the class of Werner states), but the obtained structure is compatible with previous results. To have some insight into the properties of the remaining part of the

set Λ , we modify the method of generation of states and consider density matrices lying close to the boundary of the set of all states i.e. such ρ that one of its eigenvalues is almost equal to zero. For these randomly generated states, the pairs (S_L, C) cover the whole set Λ , and its structure is the same as for the class \mathcal{E}_1 (FIG. 4). The results suggest that the picture obtained using the class \mathcal{E}_1 should be correct also for all two-qubit density matrices, although for most of randomly chosen density matrices ρ with fixed mixedness and linear entropy, either $m(\rho) > 1$ or $m(\rho) < 1$. Unfortunately, we do not know analytic description of the boundary of the enlarged region Λ_0 .

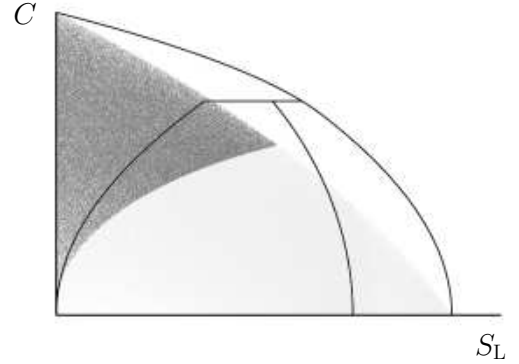


FIG. 3: The set (S_L, C) for randomly chosen two-qubit states: $m(\rho) > 1$ (dark grey), $m(\rho) < 1$ (light grey). The boundary corresponds to the family of Werner states

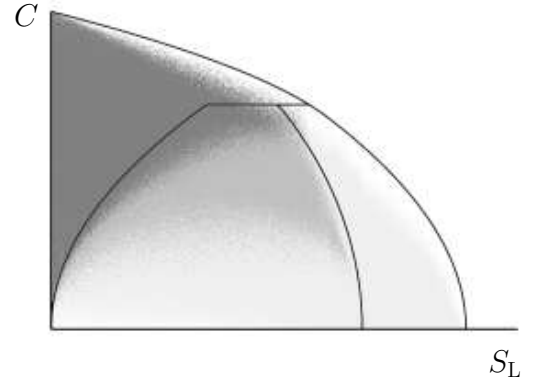


FIG. 4: The set Λ for numerically generated states lying close to the boundary: $m(\rho) > 1$ (dark grey), $m(\rho) > 1$ and $m(\rho) < 1$ (grey), $m(\rho) < 1$ (light grey)

IV. MAXIMAL CHSH VIOLATION

Consider now the values of $m(\rho)$ for states violating Bell – CHSH inequalities. We are especially interested in maximal

values of $m(\rho)$. For the class (III.1)

$$m(\rho) = \max(2c^2, (2(a+b)-1)^2 + c^2) \quad (\text{IV.1})$$

We see that (IV.1) is maximal iff $a+b=1$, and then

$$m(\rho) = 1 + c^2 \quad (\text{IV.2})$$

By general result of Verstraete and Wolf [12], for any two-qubit state, (IV.2) is the maximal degree of CHSH violation for given concurrence c . But we ask another question: what is the maximum of (IV.2) for fixed linear entropy, and which states realize that maximum? We can simply answer this question for the class of states (III.1). Since

$$S_L(\rho) = \frac{4}{3} \left(1 - a^2 - (1-a)^2 - \frac{c^2}{2} \right) \quad (\text{IV.3})$$

so fixing $S_L(\rho) = s$, we obtain

$$m(\rho) = 1 + 4(a - a^2) - \frac{3}{2}s \quad (\text{IV.4})$$

Maximum of (IV.4) is achieved at $a = \frac{1}{2}$ and equals to $2 - \frac{3}{2}s$ for $s \in [0, \frac{2}{3}]$. In this way we obtain

Theorem 1 *In the class (III.1), states maximizing degree of violation of CHSH inequalities for fixed linear entropy, lie on the curve*

$$s = \frac{2}{3} (1 - c^2)$$

and have the form

$$\rho_{\text{MVB}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \sqrt{\beta-1} e^{i\theta} & 0 \\ 0 & \sqrt{\beta-1} e^{-i\theta} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{IV.5})$$

with

$$\beta \in [1, 2], \theta \in [0, 2\pi]$$

Moreover,

$$m(\rho_{\text{MVB}}) = \beta$$

It is instructive to compare the value of $m(\rho_{\text{MVB}})$ with degree of CHSH violation for some other classes of states. Let W be the family of Werner states

$$W = (1-p) \frac{\mathbb{I}_4}{4} + p |\Psi^-\rangle \langle \Psi^-| \quad (\text{IV.6})$$

where Ψ^- is a singlet state of two-qubits. Then

$$m(W) = 2 - 2s \quad (\text{IV.7})$$

For maximally entangled mixed states ρ_{MEMS} introduced in Ref. [1], the corresponding value of m is given by

$$m(\rho_{\text{MEMS}}) = 1 - \frac{3}{4}s + \sqrt{1 - \frac{3}{2}s} \quad (\text{IV.8})$$

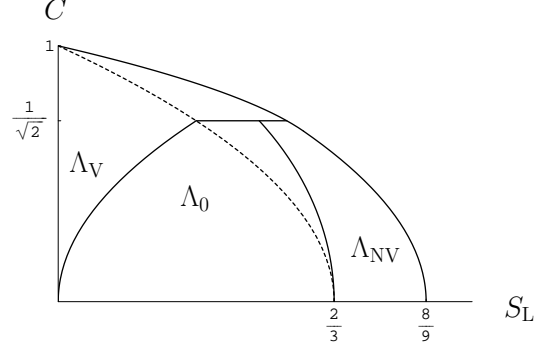


FIG. 5: States with maximal degree of CHSH violation (dotted curve) on (S_L, C) plane

We see that for a fixed linear entropy

$$m(\rho_{\text{MVB}}) \geq m(\rho_{\text{MEMS}}) \geq m(W) \quad (\text{IV.9})$$

although

$$C(\rho_{\text{MEMS}}) \geq C(W) \geq C(\rho_{\text{MVB}}) \quad (\text{IV.10})$$

So

Corollary 1 *The states with maximum amount of entanglement for a given linear entropy do not maximize degree of Bell – CHSH violation.*

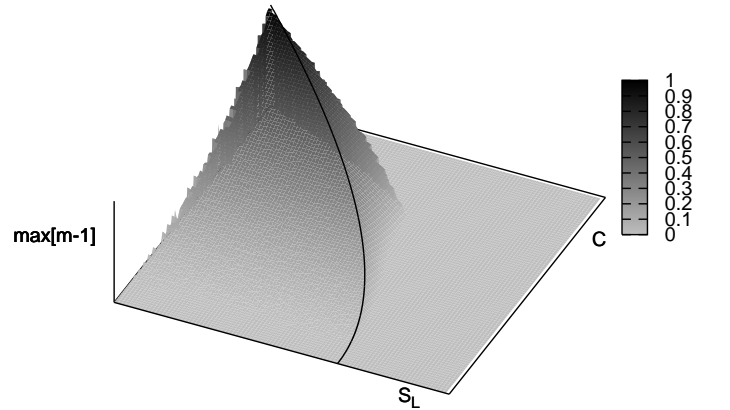


FIG. 6: Plot of $\max(m(\rho) - 1)$ as the function of S_L and C for the class \mathcal{E}_1 . The curve corresponds to $m(\rho_{\text{MVB}}) - 1$

The family of states with maximal degree of violation of Bell – CHSH inequalities has another remarkable property. It is known that fidelity of state ρ defined as

$$F(\rho) = \max \langle \psi, \rho \psi \rangle \quad (\text{IV.11})$$

where the maximum is taken over all maximally entangled pure states ψ is bounded above by [14]

$$F(\rho) \leq \frac{1 + C(\rho)}{2} \quad (\text{IV.12})$$

By direct computation, one can check that

$$F(\rho_{\text{MVB}}) = \frac{1 + C(\rho_{\text{MVB}})}{2}$$

Thus

Corollary 2 *The states ρ_{MVB} maximize fidelity for given concurrence.*

For a larger class \mathcal{E}_1 we have studied $m(\rho)$ as a function of S_L and C numerically. Again the results agree with those ob-

tained analytically for the class \mathcal{E}_0 (FIG. 6).

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